Hence Eq. (5) becomes

$$\alpha_f' = -\beta' \quad \text{if} \quad \bar{\beta} > 0$$
$$= 0 \quad \text{otherwise} \tag{6}$$

Equation (6) is the linearized unsteady flame surface approximation result in complex notation and is exact outside the zone of flame oscillation.

Acknowledgment

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References

[1] Williams, F. A., *Combustion Theory*, Addison-Wesley Publishing Co., Inc., Reading, Mass., 1965, Chaps. 1 and 3.

Buckling of Thin Plates Using the Collocation Least-Square Method

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Introduction

THE buckling behavior of thin plate elements forming webs and flanges of spars, ribs, and other portions of flight vehicles are of prime importance in the aerospace industry. Traditionally, the stability of thin plates has been analyzed by the finite difference and Rayleigh-Ritz methods.[1,2] In this Note, a relatively new numerical approach utilizing the collocation least-square concept is used to obtain a solution for buckling of clamped rectangular plates subjected to in-plane normal loads. The success of the collocation least-square method used in the present Note has already been demonstrated previously in applications to various linear and nonlinear boundary-value problems in plate bending.[3,4]

Analysis

The conventional method of collocation was first discussed in a report by Frazer et al.[5] To illustrate the method, consider a linear eigenvalue problem defined by the differential equation

$$L(W) - \lambda M(W) = 0 \tag{1}$$

where L and M are differential operators, and λ the eigenvalue. The approximate solution to Eq. (1) can be expressed as

$$W = \sum_{j=1}^{n} a_j u_j(x,y) \tag{2}$$

where a_j are undetermined coefficients and u_j are independent functions.

In conventional interior collocation, W would satisfy the prescribed boundary conditions but not the governing differential equation. The parameters $a_1,\ldots,a_n$ and the eigenvalue are evaluated by requiring the residual function, resulting from the substitution of W into Eq. (1), to be zero at n arbitrarily chosen collocation points inside the region defined by the problem. As pointed out by Collatz,[6] the choice of location of collocation points required by such a procedure is a matter of some uncertainty and the results obtained can fluctuate greatly for arbitrary choices of collocation points. This fluctuation of results can be reduced by the application of the collocation least-square concept. In applying this method, the residual function is set up at m collocation points, where $m \gg n$. This results in a system of m simultaneous equations expressed in matrix notation as:

$$[L - \lambda M]\{a\} = \{r\} \tag{3}$$

or

$$[C]\{a\} = \{r\} \tag{4}$$

where $[L]$ and $[M]$ are $m \times n$ matrices, $\{a\}$ the $n \times 1$ vector of unknown coefficients, and $\{r\}$ the $m \times 1$ vector of associated errors at m collocation points,

A least-square solution to the resulting set of equations would be that which yields a minimum value of E.

$$E = \sum_{i=1}^{m} r_i^2 = \{r\}^T\{r\} = \{a\}^T[C^TC]\{a\} \tag{5}$$

There are two alternative least-square procedures to minimize the error function E. E can be differentiated with respect to the n parameters, $(a_1, a_2,\ldots,a_n)$ resulting in $[C^TC]\{a\} = 0$, and λ is determined by setting the determinant of $[C^TC]$ equal to zero. Since the elements of $[C^TC]$ depend on λ and λ^2, solutions of the characteristic equation may take on complex values even when the actual eigenvalues for the problem are real. However, when the eigenvalues are known to be real, the real part of the complex results may furnish a good approximation.

The alternative least-sqaure scheme begins with imposing the normalization condition, which corresponds to choosing the value of one parameter, e.g., $a_1 = 1$. Then the error function E is differentiated with respect to a different set of parameters.

$$(\{a\}^j, \lambda^j) = (a_2^j, a_3^j, \ldots, a_n^j, \lambda^j)$$

where λ^j represents the jth eigenvalues associated with the jth eigenfunction, which is approximated by $W = (x,y,1,a_2^j,a_3^j,\ldots,a_n^j)$. In this approach, the error function E is treated as a nonlinear function of the eigenvalue λ, having multiple local minima, each of which corresponds to a distinct eigenvalue λ^j.

Rewriting

$$E = \{a\}^T[L - \lambda M]^T[L - M]\{a\} = \{a\}^T[P - 2\lambda Q + \lambda^2 S]\{a\} \tag{6}$$

where $P = [L]^R[L]$, $Q = ½[L^TM + M^TL]$, and $[S] = [M]^T[M]$, and setting partial derivatives of E with respect to λ and $\{a\}$ equal to zero leads to the coupled equations:

$$\lambda = \{a\}^T[Q]\{a\}/\{a\}^T[S]\{a\} \tag{7}$$

$$[P - 2\lambda Q + \lambda^2 S]\{a\} = 0 \tag{8}$$

The iteration scheme starts with an approximate value of λ to calculate $\{a\}$ from Eq. (8), then Eq. (7) is used to find an improved value of λ, and the process continues to results of desired accuracy.

While the least-square augmented collocation method formulated in this Note can be applied to a wide variety of structural buckling problems, for the sake of brevity, only the solution of a clamped rectangular plate is illustrated.

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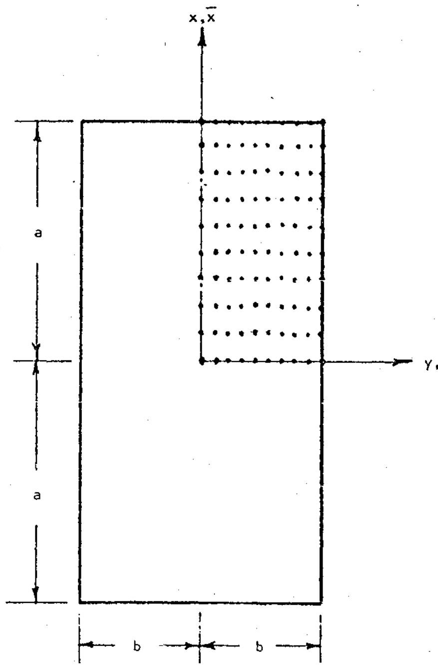

Fig. 1 Coordinate system and distribution of collocation points for clamped rectangular plates.

For a clamped rectangular plate with the coordinate systems shown in Fig. 1, subjected to in-plane compressive forces in the x and y directions, the governing differential equation can be expressed in dimensionless form as[7]:

$$\bar{W},\bar{x}\bar{x}\bar{x}\bar{x}+2R^2\bar{W},\bar{x}\bar{x}\bar{y}\bar{y}+R^4\bar{W},\bar{y}\bar{y}\bar{y}\bar{y} - N\bar{x}\bar{W},\bar{x}\bar{x}-R^2N\bar{y}\bar{W},\bar{y}\bar{y}=0 \tag{9}$$

Equation (9) is to be solved for various combinations of $N\bar{x}$ and $N\bar{y}$. Solutions are obtained for values of $R=a/b$ ranging between 1 and 2, using 100 collocation points with the distribution pattern shown in Fig. 1.

An approximate solution to Eq. (9) satisfying the boundary conditions of the clamped rectangular plate can be taken in the form

$$\bar{W}=(1-\bar{x}^2)^2(1-\bar{y}^2)^2 f_k(\bar{x},\bar{y}) \tag{10}$$

Under the action of biaxial compression, the square plate is assumed to buckle in one half-wave in both the z and y directions. As the aspect ratio a/b increases, the rectangular plate continues to buckle in one half-wave in the short y direction but it may buckle in several half-waves in the long x direction. Hence, to approximate these buckled shapes of the plate, expressions are chosen for the function f_k in Eq. (10) as

1) for modes symmetric about both the x and y axes:

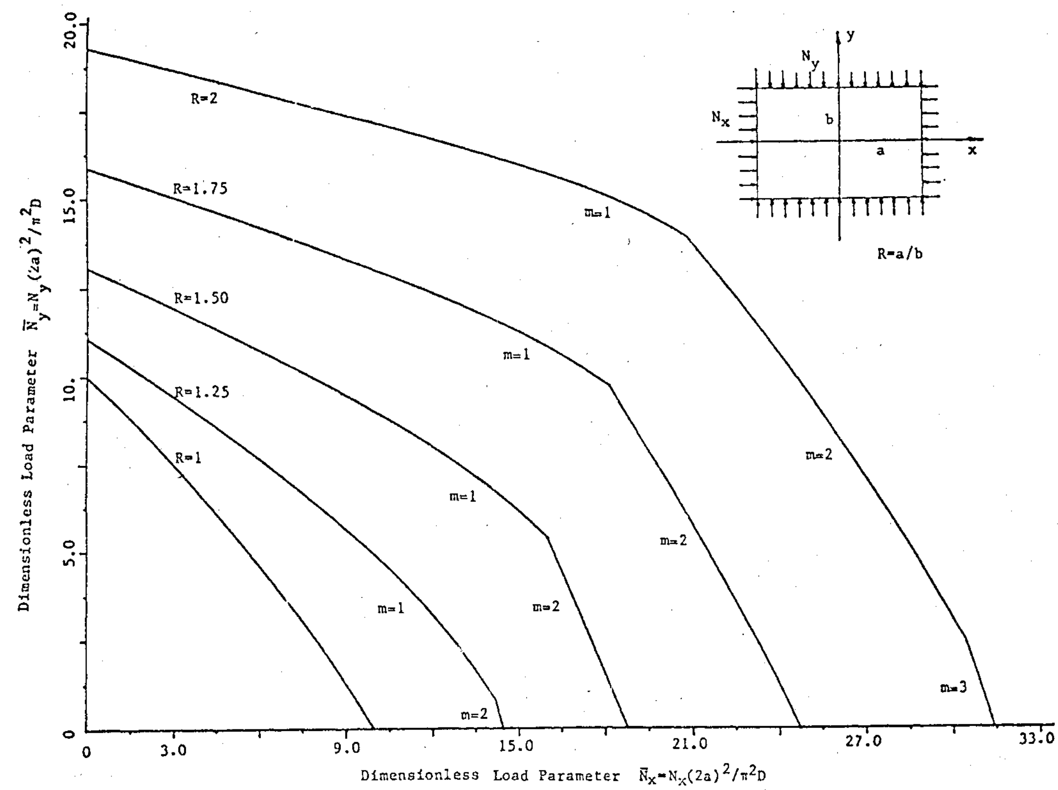

Fig. 2 Interaction curves of buckling loads for clamped rectangular plates.

Table 1 Buckling load coefficient C_r for clamped rectangular plates; $N_x = C_r(\pi^2 D/4a^2)$, $R = a/b$

N_y/N_x \ R	8	4	2	1	1/2	1/4	1/8	0 Present solution	0 Levy[9]	0 Maulbetsch[10]
	Present solution									
1.0	1.1386	2.0847 2.13[8]	3.5315 3.56[8]	5.3333 5.33[8]	7.0629 7.11[8]	8.3389 8.53[8]	9.1085	9.9565	10.07	10.48
1.25	1.2964	2.4350	4.3247	6.8472	9.9645	12.2614	13.2813	14.9959	14.45	14.65
1.50	1.5620	2.9867	5.4781	9.3258	13.9318	16.6518	17.6787	18.8017	18.74	19.01
1.75	1.9174	3.7093	6.9519	12.2607	18.4810	21.3018	22.9433	24.7119	24.84	25.02
2.0	2.3512	4.5844	8.7212	15.7692	23.1184	27.2364	29.5328	31.4260	31.52	32.24

$$f_1(\bar{x},\bar{y}) = A_{00} + A_{20}\bar{x}^2 + A_{02}\bar{y}^2 + A_{40}\bar{x}^4 + A_{22}\bar{x}^2\bar{y}^2$$
$$+ A_{04}\bar{y}^2 + A_{42}\bar{x}^4\bar{y}^2 + A_{24}\bar{x}^2\bar{y}^4 + A_{44}\bar{x}^4\bar{y}^4 \qquad (11)$$

2) for modes antisymmetric about the x axis and symmetric about the y axis:

$$f_3(\bar{x},\bar{y}) = A_{10}\bar{x} + A_{30}\bar{x}^3 + A_{12}\bar{x}\bar{y}^2 + A_{50}\bar{x}^5\bar{y} + A_{32}\bar{x}^3\bar{y}^2$$
$$+ A_{14}\bar{x}\bar{y}^4 + A_{52}\bar{x}^5\bar{y}^2 + A_{34}\bar{x}^3\bar{y}^4 + A_{54}\bar{x}^5\bar{y}^4 \qquad (12)$$

The actual solution to the buckling problem is the one of these two expressions which yields lower values for $N\bar{x}$ and $N\bar{y}$. Since only the lowest eigenvalue of Eq. (9) that corresponds to the critical load is of interest, the iterative collocation least-square scheme is the most appropriate. The iteration process begins with $N = N\bar{x} = rN\bar{y} = 0$, where r is the ratio $N\bar{x}/N\bar{y}$ and the iteration ceases when

$$|N_n - N_{n-1}| / |N_n| = 10^{-5} \qquad (13)$$

where N_n is the eigenvalue after n cycles of iteration. In all cases, this accuracy is reached within five iterations.

Results and Conclusions

Results for the buckling of clamped rectangular plates using this analysis are tabulated in Table 1. Comparisons of results are made with values obtained by Timoshenko and Gere[8] for the case of biaxial compression and by Levy[9] and Maulbetsch[10] for the case of uniaxial compression. As can be seen in Table 1, the present results are in excellent agreement with the accurate Fourier series solution reported by Levy.[9] The Ritz solutions reported by Maulbetsch[10] and Timoshenko and Gere[8] appear to be the upper bound of the maximum percentage difference between these values and the present result is less than 2.5%. Figure 2 shows plots of simultaneous critical buckling loads $N\bar{x}$ and $N\bar{y}$ for aspect ratios $R = a/b$ ranging rom 1 to 2. These curves are called interaction curves. It can be seen from the figure that the point of intersection of an interaction curve with the x axis gives the critical value of Nx for the case where $Ny = 0$. The intersection of the same curve with the y axis gives the critical value of Ny when $Nx = 0$. For the case $Nx = Ny = N_0$, the critical buckling load N_0 is determined by the intersection of these curves with the line which goes through the origin 0 of the coordinate system and makes an angle of 45 deg with the horizontal axis.

The buckled shape of the plate under a given combination of Nx and Ny for a particular value of a/b is also indicated in Fig. 2 where m is the number of half-waves in the x direction and n is the number of half-waves in the y direction.

From the results presented herein, it can be concluded that the distinct advantage of the least-square augmented collocation method lies in the fact that it completely eliminates the need for the tedious process of integration generally associated with the Galerkin or Ritz type of solution. Furthermore, in addition to its simplicity in mathematical formulation, the collocation least-square method has been demonstrated to yield accurate results that converge rapidly and such results are independent of the location and distribution of collocation points.

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References

[1] Gerard, G. and Becker, H., "Handbook of Structural Stability, Pt. I: Buckling of Flat Plates," NACA TN 3781, 1957.

[2] Flugge, W., *Handbook of Engineering Mechanics*, McGraw-Hill Book Co., N.Y., 1962.

[3] Ng, S. F., "A Collocation Least Square Solution of Boundary Value Problems in Applied Mechanics," *Proceedings of the Symposium on Computer Aided Engineering*, University of Waterloo, Canada, May 1971, pp. 395-402.

[4] Ng, S. F. and Chan, M.Y.T., "Solution of Some Boundary Value Problems in Applied Mechanics by the Collocation Least Square Method," *Journal of Computer Methods in Applied Mechanics and Engineering*, Vol. 11, No. 2, May 1977, pp. 137-150.

[5] Frazer, R. A., Jones, W. P., and Skan, S. W., "Approximation of Function and to the Solution of Differential Equations," Aerodynamics Dept. N.P.L., R&M 1799, March 1937.

[6] Collatz, L., *The Numerical Treatment of Differential Equations*, Springer, Berlin, 1960, pp. 409-413.

[7] Timoshenko, S. and Woinowsky-Krieger, S., *Theory of Plates and Shells*, McGraw-Hill Book Co., N.Y., 1959.

[8] Timoshenko, S. and Gere, J., *Theory of Elastic Stability*, 2nd ed., McGraw-Hill Book Co., N.Y., 1961.

[9] Levy, S., "Buckling of Rectangular Plates with Built-In Edges," *Journal of Applied Mechanics*, Vol. 9, Dec. 1942, pp. 171-174.

[10] Maulbetsch, J. L., "Buckling of Compressed Rectangular Plates with Built-In Edges," *Journal of Applied Mechanics*, Vol. 59, 1937, p. 59.

Influence of Suction on the Developing Wall Flow of an Impinging Jet

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Introduction

ALTHOUGH impinging jet flows have been studied extensively, little information exists on the effects of applying moderate suction along the surface of the developing flows.[1,2] The imposition of suction along the impingement surface is an inherent feature of any drying installation utilizing a combination of impingement and through flow—a process whereby an impinging hot gas is drawn through the moist permeable web by means of suction. This work stems from the need to obtain experimental information on the flowfield that would be helpful in interpretation of the role of suction on impingement heat transfer.[3]

Experimental Facility and Procedures

The room temperature uniform jet issuing from a 20 mm diam (d_n) contoured inlet nozzle impinged concentrically on a 0.97 m diam plate of which the flush-mounted central section, 348 mm diam and 9.5 mm thick, was of porous Tegraglas. The nozzle, machined according to ASME standards, had elliptical inlet and square-edged exit sections. Tegraglas porous (product of 3M Company) plate was chosen because of its excellent surface smoothness and uniform permeability. The outer ring of plexiglass served as an extended surface for jet flow away from the test section. The permeable test plate was mounted in a suction box with its top surface flush with the edges of the suction box. The small clearance between the walls of the suction box and the end of the test plate was filled

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